Dirac quantization, Virasoro and Kac–Moody algebras and boson–fermion correspondence in 2N dimensions

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ABSTRACT: Applications of Dirac's second quantization procedure to the representation theory of Virasoro and Kac-Moody algebras are discussed. An abstract picture of boson-fermion correspondence and a concretization in even space-time dimension is sketched.

1. INTRODUCTION

The viewpoints and results to be summarized here are detailed in Carey and Ruijsenaars (1987) and in Ruijsenaars (1986, 1988). A more geometric approach to the matters at issue can be found in Pressley and Segal (1986).

After recalling fermion second quantization and Dirac's twisted version of it in general terms, we indicate how it can be exploited to construct representations of Virasoro and Kac-Moody algebras. These representations are tied up with boson-fermion correspondence in two dimensions. We present an abstract picture of this correspondence, and sketch how it can be made concrete in 2N-dimensional Minkowski space-time.

2. SECOND QUANTIZATION FOR FERMIONS

Let $\mathcal{H}$ be a Hilbert space whose unit vectors are physically interpreted as states of a fermion. The fermion Fock space $\mathcal{F}_a(\mathcal{H})$ then accommodates states of an arbitrary number of fermions. The natural vehicle for getting around in $\mathcal{F}_a(\mathcal{H})$ are the creation and annihilation operators $c^{(*)}(\mathrm{v})$, $\mathrm{v}\in\mathcal{H}$, which satisfy the CAR. One-fermion operators $A\in\mathcal{L}(\mathcal{H})$ can be transported to $\mathcal{F}_a(\mathcal{H})$ as sum operators $d\Gamma(A)$ or as product operators $\Gamma(A)$, related by $\Gamma(e^A)=e^{d\Gamma(A)}$. Since Γ preserves products and $d\Gamma$ preserves commutators, one can use Γ and $d\Gamma$ to construct faithful group and Lie algebra representations, resp.

3. DIRAC'S SECOND QUANTIZATION

The above framework is adequate and useful in a nonrelativistic context. However, it does not get rid of the unphysical negative energies associated with the relativistic single particle Dirac operator. Dirac's solution to this problem ('filling the Dirac sea') can be formulated as follows. View $\mathcal{H}$ as an L^2-space on which the Dirac operator D is diagonal, and set $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$, with $\mathcal{H}_+/\mathcal{H}_-$ the positive/negative energy subspace. Then the smeared free Dirac field $\Phi(\mathrm{v})=c(P_+\mathrm{v})+c^*(\overline{P_-\mathrm{v}})$ on $\mathcal{F}_a(\mathcal{H})$ yields a unitarily inequivalent CAR representation, in which the time evolution generated via e^{itD} can be implemented by a unitary group $\tilde{\Gamma}(e^{itD})=e^{itd\tilde{\Gamma}(D)}$, with $d\tilde{\Gamma}(D)$ the (now positive) many fermion and antifermion second-quantized Dirac Hamiltonian.

More generally, CAR automorphisms $\Phi^*(\mathrm{v})\rightarrow\Phi^*(U\mathrm{v})$, U unitary, (Bogoliubov transformations) can be implemented by a unitary $\tilde{\Gamma}(U)$, provided the off-diagonal parts $U_{\pm\mp}$ are Hilbert-Schmidt (HS). Similarly, there exists an operator $d\tilde{\Gamma}(A)$ satisfying $[d\tilde{\Gamma}(A),\Phi^{*}(v)] = \Phi^{*}(Av)$ provided $A_{\pm\mp}$ are HS. Fixing the arbitrary additive constant in $d\tilde{\Gamma}(A)$ such that its vacuum expectation value vanishes, one obtains the abstract current algebra

$$[d\tilde{\Gamma}(A),d\tilde{\Gamma}(B)] = d\tilde{\Gamma}([A,B]) + C(A,B)\mathbf{1} \qquad (1)$$

$$C(A,B) \equiv \mathrm{Tr}(A_{-+}B_{+-} - B_{-+}A_{+-}).$$

Thus, one can exploit $\tilde{\Gamma}$ and $d\tilde{\Gamma}$ to construct projective representations of groups and Lie algebras of operators on the one-particle space $\mathcal{H}$, provided these operators have HS off-diagonal parts.

The framework just sketched arises in the description of charged particles. For neutral particles one needs $\mathcal{F}_a(\mathcal{H}_+)$ and the smeared Dirac-Majorana field. This yields operations $\hat{\Gamma}$ and $d\hat{\Gamma}$ that are well behaved only if the arguments satisfy a reality condition. Moreover, the neutral current algebra has a Schwinger term $C(A,B)\mathbf{1}/2$.

4. REPRESENTATIONS OF VIRASORO AND KAC-MOODY ALGEBRAS

The Virasoro and (affine) Kac-Moody algebras can be viewed as central extensions of the polynomial vector fields on the circle and of loop algebras, resp. These extensions are universal, so that one need only construct representations of the latter algebras on $\mathcal{H}$ that satisfy the HS condition to be guaranteed of representations of the former on $\mathcal{F}_a(\mathcal{H})$ via $d\tilde{\Gamma}$, and on $\mathcal{F}_a(\mathcal{H}_+)$ via $d\hat{\Gamma}$ provided the reality condition is satisfied as well. By calculating the cocycle $C(A,B)$ one can read off the (c,h)-value and level of the Virasoro and Kac-Moody representation thus obtained, resp. In this way one can get the wedge $(c,h) = (1+12\alpha,(\alpha+\beta)/2)$, $\alpha,\beta\in[0,\infty)$, in the moduli region $\{c\geqslant 1, h\geqslant 0\}$ and the $(1/2,0)$, $(1/2,1/2)$ ('Neveu-Schwarz') and $(1/2,1/16)$ ('Ramond') representations of the FQS discrete series; for the former $d\tilde{\Gamma}$ works, for the latter one needs $d\hat{\Gamma}$. In the Kac-Moody case one can obtain the basic and fundamental representations of $A_N^{(1)}$ via $d\tilde{\Gamma}$, and the basic and a fundamental representation of $B_N^{(1)}$ and $D_N^{(1)}$ via $d\hat{\Gamma}$.

All of these representations have been known for quite a while. However, the above method yields them with a minimum of effort and is, moreover, analytically clean. The same point of view has also led to new representations of Kac-Moody algebras, by using the massive Dirac operator instead of the massless one; the key point is that the Schwinger term turns out not to depend on m. In contrast to the $m = 0$ case, where type I_∞ factors arise, one obtains hyperfinite type III_1 factors for $m>0$. A rigorous version of the 'bosons' $\rightarrow$ fermions half of the boson-fermion correspondence in $2D$ has been the main tool for the structure analysis of the various representations, cf. Carey and Ruijsenaars (1987).

5. FERMIONS $\rightarrow$ BOSONS

Let $V\subset\mathcal{L}(\mathcal{H})$ be a real vector space consisting of commuting self-adjoint operators with off-diagonal HS parts. Then the current algebra (1) reduces to

$$[d\tilde{\Gamma}(A),d\tilde{\Gamma}(B)] = C(A,B)\mathbf{1}, \quad \forall A,B\in V. \qquad (2)$$

Thus, if $C(\cdot,\cdot)$ is nondegenerate on V, one gets a representation of the CCR associated with $<V,C(\cdot,\cdot)>$. This is how bosons can arise in a fermionic context.

Of course, in the abstract picture just sketched it is not at all clear such V exist, since nondegeneracy of C seems incompatible with the HS condition. However, this situation does occur for chiral gauge transformations in the one-particle Dirac theory. Specifically, in $2D$ one can set

$$\check{D} \equiv \begin{bmatrix} -i\dfrac{d}{dx} & m \\ m & i\dfrac{d}{dx} \end{bmatrix}, \quad \check{V} \equiv \left\{ \begin{bmatrix} f(x) & 0 \\ 0 & g(x) \end{bmatrix} \middle| f,g \in S_{\mathbf{R}}(\mathbf{R}) \right\} \tag{3}$$

Then V is the transform of $\check{V}$ to the spectral representation space $\mathcal{K} = L^2(\mathbf{R},dp)^2$, on which

$$D = \begin{bmatrix} E_p & 0 \\ 0 & -E_p \end{bmatrix}, \quad E_p \equiv (p^2+m^2)^{1/2} \tag{4}$$

The operators in $d\tilde{\Gamma}(V)$ are then just the smeared time-zero Dirac currents. For $m = 0$ the resulting CCR representation is the Fock representation. However, for $m > 0$ the representation is not quasi-free, and not much is known concerning the generating functional $E(f)$, in spite of its explicitness. For instance, it is known that $E(tf)$, $t \in \mathbf{R}$, is real-analytic, but not whether it is entire in t.

For $D > 2$ the HS condition is violated by any non-trivial multiplication operator. Thus, the smeared Dirac currents are only quadratic forms, not operators. Formally, they still satisfy (1), $C(A,B)$ now being an 'infinite Schwinger term'.

6. 'BOSONS' → FERMIONS

As already mentioned, there exists a unitary $\tilde{\Gamma}(U)$ satisfying

$$\tilde{\Gamma}(U)\Phi^*(v) = \Phi^*(Uv)\tilde{\Gamma}(U), \quad \forall v \in \mathcal{K} \tag{5}$$

if (and only if) U is unitary and $U_{\pm\mp}$ are HS. In particular, if $A = A^* \in \mathcal{L}(\mathcal{K})$ and $A_{\pm\mp}$ are HS, then $U = e^{itA}$ satisfies the requirements, and one can take $\tilde{\Gamma}(e^{itA}) = e^{itd\tilde{\Gamma}(A)}$. Then (5) is an abstract version of the physicist's saying: 'The Dirac currents generate the gauge transformations of the Dirac field'. It is easily seen that in this case U_{--} is Fredholm and has vanishing index. The implementer $\tilde{\Gamma}(U)$ then leaves the charge sectors invariant.

However, now assume U is unitary and satisfies

$$U_{--} \text{ Fredholm, } \operatorname{Ker} U_{--} = \{\lambda e_-\}, \ \operatorname{Ker} U_{--}^* = \{0\}. \tag{6}$$

Then the implementer can be taken to be

$$\tilde{\Gamma}(U) = c^*(e_+)E + Ec(\bar{e}_-), \quad e_+ \equiv Ue_-. \tag{7}$$

Here, E can be written down explicitly, but we shall not do so. We do need to know that when $U_{\pm\mp}$ are HS, E is an operator that leaves the charge sectors invariant; hence, $\tilde{\Gamma}(U)$ raises charge by one unit. If the HS condition is violated, one can still make rigorous sense of (5) and (7) in terms of quadratic forms. Moreover, assuming henceforth that U_ϵ is a family of unitaries satisfying (6) for any $\epsilon > 0$, and also

$$s\text{-}\lim_{\epsilon \to 0} U_\epsilon = -\mathbf{1} \tag{8}$$

(and a further technical condition), then one can infer $\lim_{\epsilon \to 0} E_\epsilon = \mathbf{1}$ in form sense. Therefore, one expects to get the Dirac field '$c^*(e_{0,+}) + c(\bar{e}_{0,-})$' by taking ϵ to 0 in $\tilde{\Gamma}(U_\epsilon)$.

This is an abstract picture for the 'bosons' → fermions part of the boson-fermion correspondence. It is mathematically deficient, since it follows from (8) that $e_{\epsilon,\pm}$ cannot have non-trivial limits $e_{0,\pm}$ in $\mathcal{K}$, but this can be taken care of in the concrete cases to be described presently. Moreover, the physical picture suggested by the terminology needs explanation (and an act of faith): The current standpoint in particle physics/string theory appears to be that the Dirac currents 'are' bosons and that an object like $\tilde{\Gamma}(U_\epsilon)$ should be viewed as creating 'coherent states' of these bosons from the vacuum. In fact, in $2D$ there is an explicit formula for the Dirac field in terms of boson fields.

As before, in the abstract context it is far from clear that families with the above properties exist. However, in $2D$ one can take the transforms to $\mathcal{H}$ of the family of chiral gauge transformations

$$\check{U}_{\epsilon,\alpha}(x) = \begin{bmatrix} \frac{i\epsilon-(x-\alpha)}{i\epsilon+(x-\alpha)} & 0 \\ 0 & -1 \end{bmatrix}, \quad \epsilon>0,\ \alpha\in\mathbf{R}. \tag{9}$$

(Note the upper slot has winding number one.) With a suitable choice of multiplicative constant one can then prove

$$\lim_{\epsilon\to 0} \tilde{\Gamma}(U_{\epsilon,\alpha}) = \psi^*_{\mathrm{Dirac}}(0,\alpha)_+ . \tag{10}$$

a relation which after smearing holds strongly on a dense subspace of $\mathcal{F}_u(\mathcal{H})$, cf. Carey and Ruijsenaars (1987).

More generally, such families can be proved to exist in even-dimensional Minkowski space-time, cf. Ruijsenaars (1988). However, one must now take recourse to chiral gauge transformations in a nonabelian context. Specifically, in $2ND$ one can take as position space $L^2(\mathbf{R}^{2N-1},dx)\otimes\mathbf{C}^{2n}\otimes\mathbf{C}^n$, where $n\equiv 2^{N-1}$, as Dirac operator

$$\check{D} = \begin{bmatrix} -i\sigma\cdot\nabla & m\mathbf{1}_n \\ m\mathbf{1}_n & i\sigma\cdot\nabla \end{bmatrix} \otimes \mathbf{1}_n,\ \sigma_j = \sigma_j^*,\ \sigma_j\sigma_k+\sigma_k\sigma_j = \delta_{jk},\ j,k = 1,\cdots,2N-1 \tag{11}$$

and then

$$\check{U}_{\epsilon,\alpha}(x) \equiv \begin{bmatrix} \mathbf{1}_n\otimes\frac{(-)^N i\epsilon+\sigma\cdot(x-\alpha)}{(-)^N i\epsilon-\sigma\cdot(x-\alpha)} & 0 \\ 0 & -\mathbf{1}_n\otimes\mathbf{1}_n \end{bmatrix} \tag{12}$$

has the required properties. (Note that for $N = 2$ the upper slot generates $\pi_3(SU(2)) = \mathbf{Z}$.) Moreover, an analog of (10) holds true.

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